Motivic Thom Spectrum and Algebraic Cobordism

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Review

Thom Space

For a vector bundle $\xi : E \to B$, the associated **Thom Space** $Th(\xi)$ is the homotopy pushout

$$\begin{array}{ccc} E - B & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & Th(\xi) \end{array}$$

If the bundle is equipped with a metric, the Thom space can be computed via:

$$\begin{array}{ccc} S(E) & \longrightarrow & D(E) \\ \downarrow & & \downarrow \\ * & \longrightarrow & D(E)/S(E) \cong Th(\xi) \end{array}$$

Thom Isomoprhism

Theorem (Thom Isomorphism)

Let $\xi : E \to B$ be an oriented real vector bundle of rank n. Then, there exists a **Thom Class**

$$c \in H^n(E, E_0; \mathbb{Z}) \cong \tilde{H}^n(Th(\xi); \mathbb{Z})$$

that restricts to the orientation cohomology class on each fiber. Moreover, there is an isomorphism

$$H^k(E; \mathbb{Z}) \to \tilde{H}^{k+n}(Th(\xi); \mathbb{Z})$$

for all $k \geq 0$ given by cupping with the Thom class.

Thom Spectra

Proposition

For vector bundles $\eta_1 : E_1 \to B_1$ and $\eta_2 : E_2 \to B_2$, we have

$$Th(\eta_1 \times \eta_2) \cong Th(\eta_1) \wedge Th(\eta_2)$$

Taking one bundle to be the trivial bundle over a point, we have

Corollary

Let ξ be the rank 1 trivial real bundle over base B. If E is some other vector bundle over B, then

$$Th(\xi^n \oplus E) \cong S^n \wedge Th(E)$$

For any vector bundle, the above equivalences gives us structure maps of the **Thom Spectra** associated to the vector bundle.

Universal Thom Space

We specialize to the complex case:

Fact

The universal rank n complex bundle is the tautological bundle

$$\gamma_n \to \mathrm{Gr}(n, \infty) = BU(n)$$

The bundle $\gamma_{n+1} \to BU(n+1)$ pulls back to $1 \oplus \gamma_n$ over $BU(n)$. Thomifying gives us structure maps

$$\Sigma^2 Th(\gamma_n) \to Th(\gamma_{n+1})$$

and the Thom spaces organize into the complex Thom spectrum

$$MU_{2n} := Th(\gamma_n)$$

Complex-Oriented Cohomology Theories

Given a (ring) spectrum E, we will use E^* to denote the (multiplicative) cohomology theory it represents.

Definition

A multiplicative cohomology theory E is **complex-orientable** if the homomorphism induced by inclusion

$$E^2(\mathbb{CP}^\infty) \to E^2(\mathbb{CP}^1)$$

is surjective.

The surjectivity condition is equivalent to a class $c_1^E \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that under the map

$$i^* : \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2) \cong \pi_0(E)$$

$i^* c_1^E$ is the generator for $\pi_0(E)$. We call the class c_1^E a **complex orientation** of E.

Examples

Example

Ordinary cohomology, $H\mathbb{Z}$, is complex-orientable:
$H^2(\mathbb{CP}^\infty;\mathbb{Z}) \to H^2(\mathbb{CP}^1;\mathbb{Z})$ is an isomorphism.

Example

Complex K-theory, KU, is complex-orientable: the class $\gamma_1 - 1$ in $\tilde{KU}^2(\mathbb{CP}^\infty) \cong \tilde{KU}^0(\mathbb{CP}^\infty)$ is a complex orientation since it restricts to the Bott element.

Non-example

Real K-theory is not complex-orientable since

$$\mathbb{Z} \cong \tilde{KO}(\mathbb{CP}^\infty) \to \tilde{KO}(\mathbb{CP}^1) \cong \mathbb{Z}$$

is multiplication by 2.

Some Computations

Through Atiyah-Hirzebruch spectral sequence, it is easy to compute a complex-orientable cohomology theory for $\mathbb{CP}^n$: the surjectivity criterion forces the spectral sequence to degenrate at E^2, and we have

Lemma

We have the isomorphisms

$$E^*(\mathbb{CP}^\infty) \cong (\pi_* E)[[t]]$$

$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong (\pi_* E)[[x, y]]$$

and the generator t is a complex orientation.

Example

For $E = H\mathbb{Z}$,

$$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[[c_1]]$$

, where c_1 is the first Chern class of the tautological line bundle.

Chern Class

One can think the orientation c_1^E as a generalized first Chern class of the tautological bundle over $BU(1)$. Recall that For $E = H\mathbb{Z}$, the Chern class provides a group isomorphism from the Picard group to H^2 of the base:

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

But for generalized Chern classes, this is no longer true.

Some Computations

The homotopy associative and commutative multiplication

$$m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$$

classifies tensor product of line bundles. and there is an induced map

$$m^* : (\pi_* E)[[t]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong (\pi_* E)[[x, y]]$$

so $m^* t = c_1^E(\gamma_1 \otimes \gamma_1)$ will be a formal power series of two variables over $\pi_*(E)$

Example

For $E = KU$, let γ_1 denote the universal bundle over $BU(1)$, then generating the cohomology class $t \in (\pi_*(KU))[[t]]$ represents the virtual bundle $\gamma_1 - 1$. Thus, we have

$$m^*(1 + t) = (1 + x)(1 + y) = 1 + x + y + xy$$

Formal Group Law

Definition

A (commutative, one-dimensional) **formal group law** over a (graded) commutative ring R is an element $f(x, y) \in R[[x, y]]$ that satisfies the following:

1. $f(x, 0) = f(0, x) = x$
2. $f(x, y) = f(y, x)$
3. $f(x, f(y, z)) = f(f(x, y), z)$

The three conditions correspond to identity, commutativity and associativity of a group operation. The existence of unique inverse can be deduced from condition 1 and 3.

Formal Group Law From Orientation

Proposition

The class $m^*t \in (\pi_* E)[[x, y]]$ is a formal group law over $\pi_*(E)$.

Example

The formal group law from the canonical orientation of $H\mathbb{Z}$ is the additive formal group law

$$f(x, y) = x + y$$

as Chern class is a group homomorphism.

Example

The formal group law from the canonical orientation of KU is the multiplicative formal group law

$$f(x, y) = x + y + xy$$

Representability

It is easy to see that taking the set of formal group laws over a commutative ring is a covariant functor

$$FGL : \textbf{CRing} \to \textbf{Set}$$

Proposition

The functor FGL is represented by a commutative ring L.

We could directly construct the ring L such that there is a bijection

$$\mathrm{Hom}(L, R) \cong FGL(R)$$

We will call the universal ring L the **Lazard ring**.

Lazard's Theorem

The existence of the universal ring L is trivial: one forms a big enough polynomial ring over $\mathbb{Z}$ and quotient out by the generating relations specified by the axioms. But explictly computing L is the following theorem:

Theorem (Lazard's Theorem)

he Lazard ring L is isomorphic to a graded polynomial ring

$$L \cong \mathbb{Z}[c_1, c_2, ...]$$

Universality of MU

The canonical equivalence

$$\mathbb{CP}^\infty \to MU(1)$$

equips MU with a orientation.

Theorem

MU is the universal complex oriented cohomology theory: every such spectrum E is equipped with a morphism of ring spectrum

$$MU \to E$$

that takes the canonical orientation of MU to that of E.

Quillen's Theorem

Quillen established the connection between the universal formal group law and the unversal complex oriented cohomology theory.

Theorem (Quillen)

The ring morphism $L \to \pi_*(MU)$ is an isomorphism of graded rings that classifies the canonical formal group law on MU.

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Assumptions

The standing assumption is that we will work with motivic spaces/spectra over a field k, with $char(k) = 0$. Some results will be true for positive characteristics as well, as well as over general k-schemes.

$\mathbb{P}^1$-Spectra

Let $\mathbb{P}^1$ be the motivic space pointed at ∞.

Definition

A $\mathbb{P}^1$-spectrum E is a sequence of pointed motivic spaces $\{E_n : n \in \mathbb{N}\}$ with structure maps

$$\sigma_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$$

A morphism is a sequence of maps of pointed spaces compatible with structure maps.

Example

Any pointed motivic space X gives rise to a $\mathbb{P}^1$ suspension spectrum, denoted by $\Sigma^\infty_{\mathbb{P}^1} X$.

Intuition

Theorem

There is a model structure on the category of $\mathbb{P}^1$-spectrum that presents the stable ∞-category $SH(k)$.

The reason it is preferred in todays discussion is:

1. Periodicity
2. Topological realization
3. Jardine's symmetric $\mathbb{P}^1$-spectra.

Motivic Thom Spectrum

Definition

The **motivic Thom space** of a vector bundle $\eta : E \to B$ is the homotopy (∞)-pushout

$$\begin{array}{ccc} E - B & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & Th(\eta) \end{array}$$

Example

Consider the trivial rank 1 bundle $\xi : X \times \mathbb{A}^1 \to X$. We have computed that

$$Th(\xi) = \mathbb{A}^1/\mathbb{A}^1 - \{0\} \cong \mathbb{P}^1 \cong S^1_s \wedge S^1_t$$

Properties

Proposition

Let $\eta_1 : E_1 \to B_1$ and $\eta_2 : E_2 \to B_2$ be vector bundles. For motivic Thom spaces, the product formula

$$Th(\eta_1 \times \eta_2) \cong Th(\eta_1) \wedge Th(\eta_2)$$

still holds.

To see this, it suffices to note that

$$E_1 \times E_2 - (B_1 \times B_2)$$

has $(E_1 - B_1) \times E_2 \coprod E_1 \times (E_2 - B_2)$ as a Zariski open cover.

Motivic Thom Spectrum

Recall that in algebraic geometric/motivic setting, we still have the tautological bundle

$$\gamma_{n,m} \to \mathrm{Gr}_n(\mathbb{A}^{n+m})$$

And we can still take colimits and define

$$MGL_n := \mathrm{colim}_m Th(\mathrm{Gr}_n(\mathbb{A}^{n+m}))$$

Let $i_n : \mathrm{Gr}_n(\mathbb{A}^{m+n}) \to \mathrm{Gr}_{n+1}(\mathbb{A}^{m+n+1})$ be the canonical inclusion. It still holds that

$$i_n^* \gamma_{n+1,m+1} \cong \xi \oplus \gamma_{n,m}$$

which produces structure maps

$$\sigma_n : \mathbb{P}^1 \wedge MGL_n \to MGL_{n+1}$$

after passing to the colimit.

Definition

The Motivic Thom Spectrum **MGL** is $\mathbb{P}^1$-spectrum given by $\{MGL_n, \sigma_i | i \in \mathbb{N}\}$

Ring Spectra

Definition

A **motivic ring spectrum** is a monoid in homomtopy category $\mathrm{SH}(k)$.

This is the ring spectrum in the weak sense, but that is all we need.

Remark

(Commutative) Monoids in Jardine's symmetric $\mathbb{P}^1$-spectra or in the ∞-category $\mathrm{SH}(k)$ will be (E_∞) A_∞-rings.

Monoid structures

Theorem ([6], Section 2.1)

The (symmetric) $\mathbb{P}^1$ spectrum MGL is a motivic ring spectrum.

The monoidal structure is induced by the closed embedding of Grassmannians

$$\mathrm{Gr}_n(\mathbb{A}^{mn}) \times \mathrm{Gr}_p(\mathbb{A}^{mp}) \to \mathrm{Gr}_{n+p}(\mathbb{A}^{m(n+p)})$$

by sending a two linear subspaces to their product. The assoicated bundle map induces a map of Thom spaces compatible with colimit.

Oriented Motivic Spectra

Definition

A motivic ring spectrum E is **oriented** if there is a class $c_E \in E^{2,1}(\mathbb{P}^\infty)$ that restricts to the generator in $E^{2,1}(\mathbb{P}^1)$ under the map

$$i^* : E^{2,1}(\mathbb{P}^\infty) \to E^{2,1}(\mathbb{P}^1)$$

Orientation of $H\mathbb{Z}$

Let $H\mathbb{Z}$ be the motivic spectrum that represents motivic cohomogy.

Theorem (MVW [3], Corollary 4.2)

Let X be a smooth scheme over k. Then,

$$H^{2,1}(X, \mathbb{Z}) \cong Pic(X)$$

Given a vector bundle $L \to X$, we can define $c_1(L) \in H^{2,1}(X; \mathbb{Z})$ to be the cohomology class that corresponds to the class $L \in \mathrm{Pic}(X)$. The universal class will be our orientation, and this gives rise to the additive formal group law.

Orientation of MGL

Lemma

The zero section

$$s : BGL_1 = \mathbb{P}^\infty \to MGL(1)$$

is a motivic equivalence.

We consider the closed immersion

$$i_n : \mathbb{P}^{n-1} \to \mathbb{P}^n$$

The normal bundle of the immersion is the canonical line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. The Purity isomorphism gives us

$$\mathbb{P}^n \cong \frac{\mathbb{P}^n}{\mathbb{P}^n - \mathbb{P}^{n-1}} \cong Th(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$$

Taking the colimit of both sides finishes.

Orientation of MGL

Corollary

The composition

$$\Sigma^\infty \mathbb{P}^\infty \to \Sigma^\infty MGL(1) \to \Sigma^\infty \mathbb{P}^1 \wedge MGL$$

defines the canonical orietation in $MGL^{2,1}(\mathbb{P}^\infty)$.

Projective Bundle Theorem

In order to compute the cohomology of Grassmannians, one needs the following:

Theorem (Projective Bundle Theorem)

Let X be a smooth variety and $V \to X$ a rank n vector bundle. For an oriented motivic cohomology theory E, we have

$$E^{*,*}(\mathbb{P}(V))c \cong E^{*,*}(X)[\epsilon]/\epsilon^n$$

where $\epsilon = c_1(\mathcal{O}_V(-1))$.

Computations

Theorem

Let E be an oriented motivic ring spectrum. Then,

$$E^{*,*}(Gr_n) = E^{*,*}[[c_1, c_2, ..., c_n]]$$

is the formal power series ring generated by the Chern classes.

For a proof, see [7] Theorem 2.0.7. The idea is one computes the E cohomology of flag varieties using the projective bundle formula, and then establish the isomorphism with E cohomology of Gr_n.

Motivic Thom Isomorphism

Theorem (Vezzosi [9], Theorem 3.8)

Let E be an oriented motivic ring spectrum, and $\xi : \epsilon \to X$ be a vector bundle of rank r. Then, there is a motivic Thom isomorphism

$$\Phi : E^{*,*}(X) \to E^{*+2r,*+r}(Th(\xi))$$

The proof breaks down to first defining the Thom class using the given orientation. Then, it utilizes the well-known equivalence

$$\mathbb{P}(\xi \oplus 1)/\mathbb{P}(\xi) \cong Th(\xi)$$

and its associated long exact sequence, together with the projective bundle formula to deduce the isomorphism.

Computations

The following is a direct corollary of Thom isomorphism

Corollary

Let E be an oriented motivic ring spectrum. Then, there is a canonical Thom isomorphism

$$E^{*,*}(BGL) \to E^{*,*}(MGL)$$

The only fact we have to use is

Lemma

For any $\mathbb{P}^1$ spectrum E, there is a canonical identification

$$\varinjlim \Sigma_{\mathbb{P}^1}^{\infty - i} E_i \cong E$$

Given another spectrum F, there is a Milnor exact sequence

$$0 \longrightarrow \lim{}^1 F^{p+2i-1,q+1}(E_i) \longrightarrow F^{p,q}(E) \longrightarrow \lim{}^{p+2i,q+i}(E_i) \longrightarrow$$

Universality

Theorem ([7])

MGL is the universal oriented motivic cohomology theory in the sense that

$$\{orientations\ of\ E\} = [MGL, E]_{ring}$$

The proof mostly follows in the same vein as in the classical case: one first shows that given a monoid map $MGL \to E$, the image of the canonical orientation of MGL is an orientation for E; conversely, an orientation of E produces uniquely an element in $E^{0,0}(MGL)$, and one shows that it is indeed a monoid map.

Motivic Quillen's Theorem

From the computation $E^{*,*}(\mathbb{P}^\infty) = E^{*,*}[[c_1]]$, it is clear that oriented motivic cohomology gives rise to formal group laws in the same way as in the classical case.

Theorem (Hoyois, Hopkins-Morel [1])

Let k be a field of characteristic exponent c. Then, there is a canonical equivalence of spectra

$$MGL/(c_1, c_2, ...)[\frac{1}{c}] \cong H\mathbb{Z}[\frac{1}{c}]$$

Outline of Proof

We outline the proof given in [1]. For simplicity, assume $char(k) = 0$, and let $f : MGL/(c_1, c_2, ...) \to H\mathbb{Z}$ be the map we want to show to be an equivalence.

1. show after smashing with $H\mathbb{Q}$, f becomes an equivalence. This uses results from motivic analogue of Landweber exactness proved in [5].

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4. The final step is to show MGL is $H\mathbb{Z}$-local. This utilizes Morel's **homotopy t-structure** on $\mathrm{SH}(k)$.

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Complex Cobordism

The homology theory MU_* is understood geometrically (or rather invented by) by Thom: it is the bordism ring Ω_* of complex manifolds.

A geometric interpretation of the cohomology theory MU^* is given later by Quillen in [8]:

Theorem (Quillen [8])

For a smooth manifold X, the group $MU^q(X)$ is isomorphic to the group of proper **complex oriented maps** *into X, denoted by Ω^q.*

Moreover, Quillen defined a more geometric notion of complex oriented cohomology theory: it would be a contravariant functor on the category of smooth manifolds, with proper complex oriented maps induceing Gysin homomorphisms.

Theorem (Quillen [8], Proposition 1.10)

Let h be a complex oriented cohomology theory. Given an element $a \in h(pt)$, there is a unique morphism

$$\Omega^* \to h$$

of functors commuting with the Gysin homomorphism that sends $1 \in \Omega^*$ to a.

Characteristic Class

The geometric definition still let us define Chern classes, and we can check that the product formula for line bundles will still give rise to formal group laws.

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Algebraic Cobordism

Following Quillen, Levine and Morel constructed the algebro-geometric analogue of oriented cohomology theories on the category of smooth k-schemes and algebraic cobordism in [4].

Definition (Rough)

An **oriented cohomology theory** on Sm_k is an additive functor

$$A^* : Sm_k^{op} \to CoGrR$$

such that for each projective morphism $f : Y \to X$ of relative codimension d, there is a Gysin homomorphism

$$f_* : A^*(Y) \to A^{*+d}(X)$$

that satisfies certain composition laws. Moreover, one has a good definition of projective bundle formula and Chern classes.

Universal Algebraic Cobordism

Theorem

Morel-Levine Assume k has characteristic zero. Then there exists a universal oriented cohomology theory on Sm_k, denoted by Ω^*.

By the existence of Chern classes, these cohomology theories also produce formal group laws.

Levine also proved the comparison theorem:

Theorem (Morel-Levine)

There is a canonical isomorphism of graded rings:

$$L \to \Omega^*(Spec(k))$$

Theorem (Morel-Levine)

Given an embedding $t : k \to \mathbb{C}$, there is a realization functor that induces isomorphism

$$\Omega^*(Speck) \to MU^{2*}(pt)$$

Theorem (Levine [2])

There is a canonical isomorphism of graded rings

$$\Omega^*(X) \to MGL^{2*,*}(X)$$

for all $X \in \mathcal{S}m_k$.

Examples

Example

The Chow ring functor CH is an oriented cohomology theory, and it gives rise to the additive formal group law.

Example

The grothendieck group functor K^0 is an oriented cohomology theory. It gives rise to the multiplicative formal group law.

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